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On the Convergence of Quadratic Spline Interpolants

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1. INTRODUCTION AND NOTATION

Let Δ denote a grid or partition of the unit interval $I=[0,1]$, i.e.,

$$\Delta: 0=x_0<x_1<\cdots<x_{n-1}<x_n=1 \qquad (n>0). \tag{1.1}$$

The maximal mesh length is abbreviated by

$$h=h(\Delta)=\max_{1\leqslant i\leqslant n} h_i, \qquad h_i=x_i-x_{i-1}.$$

As usual, by $\mathrm{Sp}(2,\Delta)$ we denote the space of quadratic spline functions determined by the above partition Δ. Namely, $s\in\mathrm{Sp}(2,\Delta)$ if and only if the following conditions are satisfied:

(i) In each subinterval $[x_{i-1},x_i]$ $(i=1,2,\ldots,n)$ s is a polynomial of degree 2 or less,

(ii) $s\in C^1(I)$.

Let $\|\cdot\|_\infty$ stand for the sup-norm on the interval I and let $\omega(f;\cdot)$ denote the modulus of continuity of $f\in C(I)$.

Now, when $f\in C(I)$ is a given function, we are interested in conditions

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assuring convergence of the corresponding quadratic spline interpolants to f as $h \to 0$. If the nodes of interpolation are chosen according to

$$x_0, \tfrac{1}{2}(x_0 + x_1), ..., \tfrac{1}{2}(x_{n-1} + x_n), x_n$$

then Marsden [3] shows uniform convergence for all $f \in C(I)$ (see also de Boor [1] and Kammerer, Reddien, and Varga [2]). This is even true for the more general nodes

$$x_0, \alpha_1 x_0 + (1 - \alpha_1) x_1, ..., \alpha_n x_{n-1} + (1 - \alpha_n) x_n, x_n$$

if only $1 - \beta \leqslant \alpha_i \leqslant \beta$ $(i = 1, 2, ..., n)$ with a constant $\beta \in [\frac{1}{2}, \sqrt{2}/2)$ (see Schmidt and Mettke [8]). In these cases the determination of the quadratic spline interpolants requires the solution of a tridiagonal system of linear equations.

The interpolation problem

$$\begin{aligned} s(x_i) &= f(x_i) \qquad (i = 0, 1, ..., n), \\ s'(x_0) &= m_0 \end{aligned} \tag{1.2}$$

(m_0 given real number) possesses also a unique solution $s \in \mathrm{Sp}(2, \Delta)$ and, moreover, s can be computed successively on the subintervals $[x_0, x_1]$, $[x_1, x_2]$,..., $[x_{n-1}, x_n]$, due to the formula

$$\frac{m_{i+1} + m_i}{2} = \frac{f(x_{i+1}) - f(x_i)}{h_{i+1}} \qquad (i = 0, 1, ..., n-1) \tag{1.3}$$

where $m_i = s'(x_i)$. For the splines s given by (1.2), Mettke, Pfeifer, and Neuman [5] prove the boundedness of $\|f - s\|_\infty$ provided that f is Lipschitz-continuous and the grids Δ are not so far from the equidistant partitions. Furthermore the first author of this paper derived an explicit expression for the norm of the corresponding spline interpolation projector L_Δ^2 if $m_0 = 0$, namely,

$$\|L_\Delta^2\| = 1 + \max_{1 \leqslant i \leqslant n} h_i \sum_{j=1}^{i-1} \frac{1}{h_j} \tag{1.4}$$

(see [6]). The operator norm $\|\cdot\|$ is defined as

$$\|L_\Delta^2\| = \sup\{ \|L_\Delta^2 f\|_\infty : \|f\|_\infty \leqslant 1 \}.$$

Since for any set of partitions Δ with $h \to 0$ the norms $\|L_\Delta^2\|$ are unbounded, there exist functions $f \in C(I)$ such that

$$\sup_\Delta \|f - s\|_\infty = \infty, \tag{1.5}$$

and the set of these functions f is dense in $C(I)$. This statement is an immediate consequence of the famous Banach–Steinhaus theorem for which one is referred, e.g., to Wloka [9]. The unboundedness of certain quadratic spline projectors was also established by Meinardus and Taylor [4].

The plan of the paper is as follows. In Section 2 we extend formula (1.4) to the spline operator $\tilde{L}_\Delta^2$ described by the interpolation scheme (1.2) with arbitrary m_0. It turns out that m_0 is of no relevancy with regard to the behaviour convergence of these splines. In Section 3, by means of the derivatives m_i, we derive conditions that guarantee uniform convergence of s to f as $h \to 0$. These results may also be useful if one is looking for a divergence example. In the final section we introduce a new space $\mathrm{Sp}(2, \delta)$ of the quadratic splines where now $\Delta \subset \delta$. We show that there are quadratic splines associated with the interpolation scheme (1.2) which are uniformly convergent to the interpolated function f as h tends to zero. Because the interpolation is restricted to the grid Δ these splines $s \in \mathrm{Sp}(2, \delta)$ are not uniquely determined. For further results in this direction see the report [7].

2. Norm of the Spline Operator $\tilde{L}_\Delta^2$

With regard to the interpolation scheme (1.2) we introduce the quadratic spline operator $\tilde{L}_\Delta^2 \colon C(I) \times R^1 \to \mathrm{Sp}(2, \Delta)$ in the manner

$$\begin{aligned} (\tilde{L}_\Delta^2(f, m_0))(x_i) &= f(x_i) \qquad (i = 0, 1, ..., n), \\ (\tilde{L}_\Delta^2(f, m_0))'(x_0) &= m_0, \end{aligned} \tag{2.1}$$

where $f \in C(I)$ and $m_0 \in R^1$. This is a slightly extended version of the spline projector L_Δ^2 treated in [6]. The space $C(I) \times R^1$ is endowed with the norm

$$\|(f, m_0)\|_\infty = \max\{\|f\|_\infty, |m_0|\}.$$

Thus, $C(I) \times R^1$ is a Banach space.

The main result of this section reads as follows.

THEOREM 2.1. *Let $\tilde{L}_\Delta^2$ be the spline operator uniquely determined by the interpolatory conditions* (2.1). *Then*

$$\|\tilde{L}_\Delta^2\| = 1 + \max_{1 \leqslant i \leqslant n} h_i \left(\frac{1}{4} + \sum_{j=1}^{i-1} \frac{1}{h_j}\right). \tag{2.2}$$

Proof. In order to prove (2.2) we use the so-called fundamental splines $s_j \in \mathrm{Sp}(2, \Delta)$ $(j = 0, 1, ..., n+1)$. These are such that

$$s_j(x_i) = \delta_{ij} \qquad (i = 0, 1, ..., n;\ j = 0, 1, ..., n+1)$$

where as usual δ_{ij} stands for the Kronecker symbol. Moreover,

$$s_j'(x_0) = 0 \qquad (j = 0, 1, ..., n), \qquad s_{n+1}'(x_0) = 1.$$

Any $s = \tilde{L}_\Delta^2(f, m_0)$ can be expressed in terms of the unique s_j in the manner

$$s(x) = \sum_{j=0}^{n} f(x_j)\, s_j(x) + m_0 s_{n+1}(x) \qquad (x \in I). \tag{2.3}$$

Explicit formulae for s_j $(j = 0, 1, ..., n)$ have been derived in [6],

$$\begin{aligned}
s_j(x) &= 0, && x_0 \leqslant x \leqslant x_{j-1},\\
&= \left(\frac{x - x_{j-1}}{h_j}\right)^2, && x_{j-1} \leqslant x \leqslant x_j,\\
&= 1 + \frac{2(x_{j+1} - x)(x - x_j)}{h_j h_{j+1}} - \left(\frac{x - x_j}{h_{j+1}}\right)^2, && x_j \leqslant x \leqslant x_{j+1},\\
&= (-1)^{l+1} \frac{\tilde{m}_{j+1}}{h_{j+l+1}} (x - x_{j+l})(x_{j+l+1} - x), &&\\
& && x_{j+l} \leqslant x \leqslant x_{j+l+1}\ (l \geqslant 1),
\end{aligned} \tag{2.4}$$

with

$$\tilde{m}_{j+1} = -2\left(\frac{1}{h_j} + \frac{1}{h_{j+1}}\right) \qquad \left(j = 0, 1, ..., n-1; \frac{1}{h_0} := 0\right). \tag{2.5}$$

In addition, when using (1.3) we get

$$s_{n+1}(x) = (-1)^{j+1} \frac{(x - x_{j-1})(x_j - x)}{h_j} \qquad \text{for } x_{j-1} \leqslant x \leqslant x_j. \tag{2.6}$$

It is easy to check by means of (2.3) that

$$\|\tilde{L}_\Delta^2\| = \|\tilde{\Lambda}_\Delta^2\|_\infty,$$

where

$$\tilde{\Lambda}_\Delta^2(x) = \sum_{j=0}^{n+1} |s_j(x)| \qquad (x \in I)$$

denotes the Lebesgue function of $\tilde{L}_\Delta^2$. Now taking into account (2.4), (2.5), and (2.6) we get for $x_{i-1} \leqslant x \leqslant x_i$

$$\tilde{\Lambda}_\Delta^2(x) = \sum_{j=0}^{i-2} |s_j(x)| + s_{i-1}(x) + s_i(x) + |s_{n+1}(x)|.$$

Straightforward calculations yield

$$\tilde{\Lambda}_\Delta^2(x) = 1 + \frac{4(x_i - x)(x - x_{i-1})}{h_i}\left(\frac{1}{h_1} + \frac{1}{h_2} + \cdots + \frac{1}{h_{i-1}} + \frac{1}{4}\right).$$

Hence the assertion (2.2) follows immediately. ∎

Making use of (1.4) and (2.2) we arrive at

COROLLARY 2.2. *For the spline operators* L_Δ^2 *and* $\tilde{L}_\Delta^2$ *we have*

$$\|L_\Delta^2\| \leqslant \|\tilde{L}_\Delta^2\| \leqslant \|L_\Delta^2\| + h/4. \tag{2.7}$$

Thus, the norms $\|\tilde{L}_\Delta^2\|$ are unbounded on a set of grids Δ if and only if the norms $\|L_\Delta^2\|$ are unbounded. Therefore, the Banach–Steinhaus theorem allows the conclusion (1.5), also for the spline interpolants (1.2) with $m_0 \neq 0$.

Let us close this section with the following essential

PROPOSITION 2.3. *For any set of partitions* Δ *with* $h(\Delta) \to 0$ *the norms* $\|\tilde{L}_\Delta^2\|$ *are unbounded. Therefore a function* f *in* $C(I)$ *and a number* m_0 *exist such that*

$$\sup_\Delta \|f - \tilde{L}_\Delta^2(f, m_0)\|_\infty = \infty.$$

Proof. Suppose that the norms $\|\tilde{L}_\Delta^2\|$ are bounded, i.e.,

$$\|\tilde{L}_\Delta^2\| \leqslant M.$$

In view of (2.2) this implies

$$\frac{1}{h_1} + \frac{1}{h_2} + \cdots + \frac{1}{h_{i-1}} \leqslant \frac{M}{h_i} \qquad (i = 2, 3, ..., n).$$

Thus, starting with $h_1 \leqslant h$ and proceeding by induction we obtain immediately

$$h_i \leqslant \frac{M^{i-1} h}{(1+M)^{i-2}} \qquad (i = 2, 3, ..., n).$$

Now, because of $h_1 + h_2 + \cdots + h_n = 1$ it follows that

$$1 \leqslant h + Mh\left[1 + \frac{M}{1+M} + \left(\frac{M}{1+M}\right)^2 + \cdots\right] = h(1 + M + M^2).$$

This contradicts with $h = h(\Delta) \to 0$. The proof is complete. ∎

3. Convergence Conditions by Means of the Derivatives m_i

In this section we derive sufficient convergence conditions formulated by means of the derivatives

$$m_i = s'(x_i) \qquad (i = 0, 1, ..., n) \tag{3.1}$$

of the spline interpolant $s \in \mathrm{Sp}(2, \Delta)$. We assume that s satisfies the interpolatory conditions (1.2). For $x_{i-1} \leqslant x \leqslant x_i$ the spline interpolant s may be written as

$$\begin{aligned} s(x) = r(x) + f(x_{i-1}) :&= \frac{(x - x_{i-1})^2}{2h_i} m_i \\ &+ \left(\frac{h_i}{2} - \frac{(x_i - x)^2}{2h_i}\right) m_{i-1} + f(x_{i-1}), \end{aligned} \tag{3.2}$$

where the m's satisfy the consistency relations (1.3). Now we establish a series of partial results which lead to the main result of this section. These are contained in the following three assertions.

Assertion 3.1. *Let* $|m_{i-1}| \leqslant c$, $|m_i| \leqslant c$. *Then for* $x_{i-1} \leqslant x \leqslant x_i$ *we have*

$$|f(x) - s(x)| \leqslant ch + \omega(f; h). \tag{3.3}$$

Proof. For $x_{i-1} \leqslant x \leqslant x_i$ the estimates

$$0 \leqslant \frac{(x - x_{i-1})^2}{2h_i} \leqslant \frac{h_i}{2}, \qquad 0 \leqslant \frac{h_i}{2} - \frac{(x_i - x)^2}{2h_i} \leqslant \frac{h_i}{2} \tag{3.4}$$

hold true. Thus, we get by (3.2) and (3.4)

$$|f(x) - s(x)| \leqslant |r(x)| + |f(x) - f(x_{i-1})| \leqslant ch + \omega(f; h)$$

provided $x_{i-1} \leqslant x \leqslant x_i$. ∎

ASSERTION 3.2. *Let $m_{i-1} m_i \geqslant 0$. Then for $x_{i-1} \leqslant x \leqslant x_i$ we get*

$$|f(x) - s(x)| \leqslant 2\omega(f; h). \tag{3.5}$$

Proof. Making use of (3.2), (3.4), and (1.3) one obtains

$$|f(x) - s(x)| \leqslant |r(x)| + \omega(f; h) \leqslant \frac{h_i}{2} |m_i + m_{i-1}| + \omega(f; h) \leqslant 2\omega(f; h). \quad \blacksquare$$

The following properties of the quadratic r will be employed in the sequel:

$$\begin{aligned} r(x_{i-1}) &= 0, \\ r(x_i) &= \frac{h_i}{2}(m_i + m_{i-1}) = f(x_i) - f(x_{i-1}), \\ r(\tilde{x}_i) &= \frac{h_i m_{i-1}^2}{2(m_{i-1} - m_i)} \qquad \text{for} \quad r'(\tilde{x}_i) = 0. \end{aligned} \tag{3.6}$$

Further we introduce two sets K_α and L_α, where $\alpha \geqslant 1$ is a large but fixed real. These sets are cones with vertices at the origin and they are defined in the following way:

$$K_\alpha := \left\{ (p, q) \in R^2 \colon -\sqrt{\frac{\alpha+1}{\alpha}}\, p < q < -\sqrt{\frac{\alpha-1}{\alpha}}\, p \wedge p > 0 \right\},$$

$$L_\alpha := \left\{ (p, q) \in R^2 \colon -\sqrt{\frac{\alpha-1}{\alpha}}\, p < q < -\sqrt{\frac{\alpha+1}{\alpha}}\, p \wedge p < 0 \right\}.$$

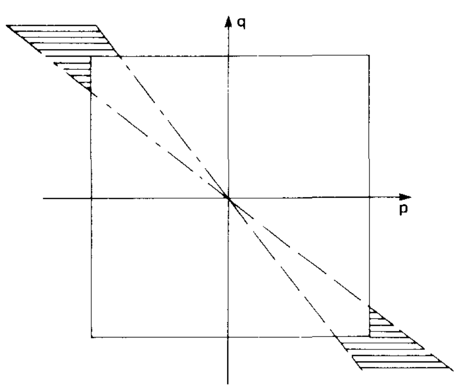

FIG. 1. The set $(K_\alpha \cup L_\alpha) \setminus Q_c$. Q_c, ——; K_α, – – –; L_α, · · · .

It is obvious that K_α and L_α lie in the fourth and second quadrants, respectively (see Fig. 1).

Now we state and prove the following

ASSERTION 3.3. *If* $m_{i-1}m_i < 0$ *and* $(m_{i-1}, m_i) \notin K_\alpha \cup L_\alpha$, *then*

$$|f(x) - s(x)| \leqslant (\alpha + 1)\,\omega(f; h) \qquad \text{for} \quad x_{i-1} \leqslant x \leqslant x_i. \tag{3.7}$$

Proof. We derive (3.7) in the case when $(m_{i-1}, m_i) \notin K_\alpha$ but lies in the fourth quadrant. Dual arguments, which we omit, give the thesis when $(m_{i-1}, m_i) \notin L_\alpha$ but lies in the second quadrant. In order to establish (3.7) in the case under consideration assume at first

$$0 > m_i \geqslant -\sqrt{\frac{\alpha - 1}{\alpha}}\, m_{i-1}. \tag{3.8}$$

Hence we get $\alpha m_i^2 \leqslant (\alpha - 1)\, m_{i-1}^2$, $m_{i-1}^2 \leqslant \alpha(m_{i-1}^2 - m_i^2)$, and

$$0 < \frac{m_{i-1}^2}{m_{i-1} - m_i} \leqslant \alpha(m_{i-1} + m_i).$$

Combining this result with (3.6) leads us to $r(\tilde{x}_i) \leqslant \alpha r(x_i)$. Since $r(x_{i-1}) = 0$, $r(x_i) > 0$, we have $0 \leqslant r(x) \leqslant r(\tilde{x}_i)$ for $x_{i-1} \leqslant x \leqslant x_i$ and in consequence

$$0 \leqslant r(x) \leqslant \alpha r(x_i) \leqslant \alpha\omega(f; h).$$

Making use of the formula (3.2) one gets the desired results in the case (3.8).

Assume next

$$m_i \leqslant -\sqrt{\frac{\alpha + 1}{\alpha}}\, m_{i-1} < 0. \tag{3.9}$$

Hence we obtain successively $\alpha m_i^2 \geqslant (\alpha + 1)\, m_{i-1}^2$, $m_{i-1}^2 \leqslant -\alpha(m_{i-1}^2 - m_i^2)$,

$$0 < \frac{m_{i-1}^2}{m_{i-1} - m_i} \leqslant -\alpha(m_{i-1} + m_i).$$

Therefore, according to (3.6), we arrive at $0 < r(\tilde{x}_i) \leqslant -\alpha r(x_i) = \alpha|r(x_i)|$. Thus $|r(x)| \leqslant \alpha|r(x_i)| \leqslant \alpha\omega(f; h)$ provided $x_{i-1} \leqslant x \leqslant x_i$. Now taking into account (3.2) we obtain the thesis in the case (3.9). This completes the proof. ∎

Before presenting the main result of this section we introduce the square

$$Q_c := \{(p, q) \in R^2 \colon |p| \leqslant c \wedge |q| \leqslant c\},$$

where c is a large but fixed real (see Fig. 1).

THEOREM 3.4. *For given* $f \in C(I)$, $m_0 \in R^1$ *let* $s \in \mathrm{Sp}(2, \Delta)$ *be the spline interpolant satisfying* (1.2), *and let* $m_i = s'(x_i)$. *If*

$$(m_{i-1}, m_i) \notin (K_\alpha \cup L_\alpha) \backslash Q_c \qquad \text{for} \quad i = 1, 2, ..., n,$$

then

$$\|f - s\|_\infty \leqslant ch + (\alpha + 1)\, \omega(f; h).$$

Proof. Combine the Assertions 3.1–3.3. ∎

4. CONVERGENCE OF SPLINES ON GRIDS WITH ADDITIONAL KNOTS

As in the previous sections, let Δ be the partition (1.1) of the unit interval I. For our purposes we introduce additional knots $t_i \in (x_{i-1}, x_i)$ and also define a new partition δ of I as

$$\delta\colon 0 = x_0 < t_1 < x_1 < \cdots < x_{n-1} < t_n < x_n = 1. \tag{4.1}$$

In this section we work with the space $\mathrm{Sp}(2, \delta)$ instead of $\mathrm{Sp}(2, \Delta)$. It should be noted that $s \in \mathrm{Sp}(2, \delta)$ is not uniquely determined by the interpolatory conditions (1.2) imposed only on Δ. Nevertheless it can be shown that there exist always splines $s \in \mathrm{Sp}(2, \delta)$ satisfying (1.2) which converge uniformly to the interpolated function $f \in C(I)$ as $h \to 0$.

To this end, we employ the B-splines M_i of order 2, where

$$M_i(x) = 2[x_{i-1}, t_i, x_i](\cdot - x)_+ \qquad (i = 1, 2, ..., n). \tag{4.2}$$

Here as usual $x_+ = \max\{0, x\}$ and $[\rho_i, \rho_{i+1}, \rho_{i+2}]\, g$ denotes the second-order divided difference of the function g at the points $\rho_i, \rho_{i+1}, \rho_{i+2}$. Further let

$$C_i(x) = \int_{-\infty}^{x} M_i(t)\, dt \qquad (i = 1, 2, ..., n). \tag{4.3}$$

For the reader's convenience we recall that

$$\begin{aligned} C_i(x) &= 0 && \text{for} \quad x \leqslant x_{i-1}, \\ &= \frac{(x - x_{i-1})^2}{(1 - \lambda_i)\, h_i^2} && \text{for} \quad x_{i-1} \leqslant x \leqslant t_i, \\ &= 1 - \frac{(x_i - x)^2}{\lambda_i h_i^2} && \text{for} \quad t_i \leqslant x \leqslant x_i, \\ &= 1 && \text{for} \quad x \geqslant x_i, \end{aligned} \tag{4.4}$$

where $\lambda_i \in (0, 1)$ is determined by $t_i = \lambda_i x_{i-1} + (1-\lambda_i) x_i$. Of course, we have $C_i \in \mathrm{Sp}(2, \delta)$ $(i = 1, 2, ..., n)$. Now we define the function s in the manner

$$s(x) = f(x_0) + (x - x_0) m_0 + \sum_{j=1}^{n} (f(x_j) - f(x_{j-1}) - h_j m_0) C_j(x) \qquad (x \in I). \tag{4.5}$$

It is obvious that $s \in \mathrm{Sp}(2, \delta)$. Further properties of this spline s are summarized below.

ASSERTION 4.1. *The spline function* $s \in \mathrm{Sp}(2, \delta)$ *defined by* (4.5) *is such that*

$$\begin{aligned} s(x_i) &= f(x_i) && (i = 0, 1, ..., n), \\ s(t_i) &= \lambda_i f(x_{i-1}) + (1 - \lambda_i) f(x_i) && (i = 1, 2, ..., n), \\ s'(x_i) &= m_0 && (i = 0, 1, ..., n), \\ s'(t_i) &= \frac{2}{h_i} (f(x_i) - f(x_{i-1})) - m_0 && (i = 1, 2, ..., n). \end{aligned} \tag{4.6}$$

Proof. Using (4.4) we get for $x_{i-1} \leqslant x \leqslant x_i$

$$s(x) = (1 - C_i(x)) f(x_{i-1}) + C_i(x) f(x_i) + (x - x_{i-1} - h_i C_i(x)) m_0. \tag{4.7}$$

By direct calculations one obtains easily the formulae (4.6). ∎

We are now ready to state the announced convergence result.

THEOREM 4.2. *Let* $f \in C(I)$. *Then for the quadratic spline* $s \in \mathrm{Sp}(2, \delta)$ *described by* (4.5) *the estimation*

$$\|f - s\|_\infty \leqslant \tfrac{1}{4} |m_0| h + \omega(f; h) \tag{4.8}$$

is valid.

Thus the above theorem says that we have a case of uniform convergence $s \to f$ as $h \to 0$ for any real m_0 and any (but fixed) $f \in C(I)$.

Proof. In view of (4.7) we get for $x_{i-1} \leqslant x \leqslant x_i$

$$\begin{aligned} f(x) - s(x) = {} & (1 - C_i(x))(f(x) - f(x_{i-1})) + C_i(x)(f(x) - f(x_i)) \\ & -(x - x_{i-1} - h_i C_i(x)) m_0. \end{aligned}$$

Therefore the statement (4.8) follows immediately because of $0 \leqslant C_i(x) \leqslant 1$, and

$$x - x_{i-1} - h_i C_i(x) = \frac{(t_i - x)(x - x_{i-1})}{t_i - x_{i-1}} \quad \text{for} \quad x_{i-1} \leqslant x \leqslant t_i,$$

$$= \frac{(x_i - x)(t_i - x)}{x_i - t_i} \quad \text{for} \quad t_i \leqslant x \leqslant x_i. \quad \blacksquare$$

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